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FORCED HEAT CONVECTION IN LINEAL FLOW OF NON-NEWTONIAN FLUIDS THROUGH RECTANGULAR CHANNELS

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Abstract—A study is made of the average rate of heat transferred to the wall of a rectangular channel when a hot incompressible non-Newtonian fluid flows through it. The fluid is that which is commonly called 'third grade' and is the Rivlin–Ericksen fluid of highest order which can yield rectilinear flow without secondary cross flow in a rectangular channel. A heat-transfer coefficient has been evaluated for several values of a non-Newtonian parameter $\bar{\varepsilon}$, and for a range of rectangular geometries from the square at one end to infinite parallel planes at the other. The results show increasing enhancement of the heat-transfer coefficient for any $\bar{\varepsilon}$ value, on moving away from the square towards narrower rectangles. An increase of the heat-transfer coefficient is also found for those fluids which exhibit large strain-rate gradients in the wall region of an arbitrary rectangular channel.

NOMENCLATURE

$\mathbf{T}$,	stress tensor;
p,	hydrostatic pressure;
$\mathbf{S}$,	determinate stress;
v_0,	average velocity;
T,	temperature field;
c_p,	specific heat at constant pressure;
k,	thermal conductivity;
Pe,	Péclet number;
d,	representative length;
$Nu(z)$,	local Nusselt number;
$\theta_M(z)$,	local meaned temperature;
$\mu, \alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3$,	material parameters;
ε,	specific driving force;
γ,	ratio of long to short side of rectangle;
ρ,	density;
ν,	body force potential;
ϕ,	$p/\rho + \nu$.

1. INTRODUCTION

THE EVALUATION of the exchange of heat between a hot Newtonian fluid and the cold wall of the channel in which it flows has been reviewed extensively in the literature. On the other hand, relatively little attention has been given to the heat-transfer problem for viscoelastic fluids, which is rather surprising in view of the many applications that it may have. Schenk and Van Laar [1] and Mahalingam *et al.* [2] for a circular tube, Chandrupatla and Sastri [3] for a square channel, have obtained increases of greater or lesser extent in the heat-transfer coefficient by using various empirical models to exhibit non-Newtonian fluid response. One of the difficulties in the way of a serious analytical approach is that for viscoelastic fluids rectilinear flow in channels, aside from those with circular cross section or consisting of a gap between infinite parallel planes, is not generally possible because it is well known that secondary flow occurs in the cross section of the channel. In recent experimental work Mena *et al.* [4] have found enhanced heat transfer for a rectangular channel compared with a circular tube, a result they attribute to secondary flow. In earlier experiments, Oliver and Karim [5] had obtained increases in the heat-transfer coefficient for flattened (circular) tubes when compared to a circular channel. These increases they attributed partly to increased tube-wall shear rate for the flattened tubes and partly to secondary flow patterns.

An extensive review of viscoelastic response can be read in [6]. In particular the fluid considered for the purposes of this report is the third order Rivlin–Ericksen fluid [7], sometimes called a fluid of third grade, which may be viewed as an approximation, to within terms of order three in the time scale, for slow motions of an incompressible simple fluid [8]. This is the fluid of highest order that is capable of sustaining rectilinear flow in an arbitrary channel without secondary transverse flow. By obviating secondary flow we hope to isolate the effect of wall shear rate on the heat-transfer coefficient for a variety of rectangular channels. We find as a general conclusion that a non-Newtonian parameter $\bar{\varepsilon}$ defined in the text, which yields increased or decreased strain-rate gradients at the channel wall according to whether it is negative or positive, leads correspondingly to enhanced or diminished heat transfer. This conclusion is in agreement with results obtained in [11] for plane Poiseuille flow between two parallel flat plates and circular tube flow. The relatively straightforward velocity profile for the plane channel, as compared to the complicated one of the current paper, is usefully recalled here to demonstrate the character of the flow.

It is

$$v = \frac{3v_0}{2K_1(\bar{\varepsilon})}\left[(1-x^2) - \frac{\bar{\varepsilon}}{2}(1-x^4)\right]; \quad |x| \le 1,$$

where the notation is that of [11]. Essentially this profile affords a basis of comparison with the empirical power-law models, e.g. in Mahalingam *et al.* [2], used to describe the flow of polymer melts in plane and circular channels. At least qualitatively for $\bar{\varepsilon} < 0$, it exhibits behaviour in common with the polyacrylamide solutions in [4] and [5] which are shear thinning fluids. The parameter $\bar{\varepsilon}$ is proportional to $\mu^{-3}(\beta_2+\beta_3)\varepsilon^2$ where ε in the absence of body forces is the (small) pressure gradient behind the flow, $\mu > 0$ is the viscosity of the fluid, and $(\beta_2+\beta_3)$ is a sum of material parameters to which we are unable to attach a preferred sign. Realistically we would not expect $\bar{\varepsilon}$ to assume other than small values consistent with the third-order fluid approximation [see equation (2.13) below]. However, it is still worth noting that the asymptotic form of v above as $|\bar{\varepsilon}| \to \infty$ is proportional to $(1 - x^4)$ which, as a power-law model, is not an unreasonable one for plane channel flow of many polymer melts. We know of no experimental results which would enable us to attribute a magnitude to $\bar{\varepsilon}$ and have therefore computed results for $\bar{\varepsilon}$ values in the range $|\bar{\varepsilon}| \le 1$, which we believe to be plausible. We have also included $\bar{\varepsilon} \to -\infty$, if only to provide bounds on the Nusselt number although, apart from this aspect of tidiness and the remark above concerning the asymptotic form of the velocity profile for plane channel flow, we doubt that these cases have any substantial validity within the third order approximation.

2. THE VELOCITY FIELD

For a homogeneous, incompressible fluid of density ρ subject to conservative specific body force $\mathbf{f} = -\text{grad}\, v$, $v(\mathbf{x}, t)$ being a scalar valued function computed on the current locations $\mathbf{x}$ of the fluid particles, the equation of linear momentum balance has the form

$$\text{div}\,\mathbf{S} - \rho\,\text{grad}\,\phi = \rho\ddot{\mathbf{x}} \tag{2.1}$$

where

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S} \tag{2.2}$$

is the stress tensor, p is an arbitrary hydrostatic pressure, and

$$\phi = \frac{p}{\rho} + v. \tag{2.3}$$

For a third-order fluid the determinate stress $\mathbf{S}$ has the form

$$\mathbf{S} = \sum_{k=1}^{3} \mathbf{S}_k,$$

$$\mathbf{S}_1 = \mu\,\mathbf{A}_1; \quad \mathbf{S}_2 = \alpha_1\,\mathbf{A}_2 + \alpha_2\,\mathbf{A}_1^2; \tag{2.4}$$

$$\mathbf{S}_3 = \beta_1\,\mathbf{A}_3 + \beta_2(\mathbf{A}_2\mathbf{A}_1 + \mathbf{A}_1\mathbf{A}_2) + \beta_3(\text{tr}\,\mathbf{A}_2)\mathbf{A}_1$$

where μ, α_1, α_2, β_1, β_2, β_3 are material parameters which we take to be constant, and $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$ are the first three Rivlin–Ericksen tensors

$$\mathbf{A}_1 = \tfrac{1}{2}(\mathbf{L} + \mathbf{L}^T);$$

$$\mathbf{A}_n = \dot{\mathbf{A}}_{n-1} + \mathbf{L}^T\mathbf{A}_{n-1} + \mathbf{A}_{n-1}\mathbf{L}, \quad n = 2, 3 \tag{2.5}$$

in which a superposed T denotes transposition and $\mathbf{L} = \text{grad}\,\dot{\mathbf{x}}$ is the velocity gradient.

A steady lineal flow in a channel of cross section R is described by

$$\dot{\mathbf{x}} = v(\mathbf{r})\mathbf{k}; \quad v(\mathbf{r}) = 0, \quad \forall \mathbf{r} \in \partial R \tag{2.6}$$

where $\mathbf{k}$ is a unit normal to the plane of R, $\mathbf{r}$ is the position of a point in this plane, and ∂R is the boundary of R. Specifically let (x_1, x_2, x_3) be orthogonal Cartesian coordinates chosen so that x_3 is in the direction of $\mathbf{k}$ and boundary ∂R is the rectangle given by $x_1 = \pm a$, $x_2 = \pm b$. The origin of coordinates is situated at the thermal entrance to the channel. It follows that equation (2.6) is compatible with the equations of motion (2.1) only if

$$\frac{\partial^2 \phi}{\partial x_3^2} = 0; \quad \frac{\partial^2 \phi}{\partial x_3 \partial x_1} = 0; \quad \frac{\partial^2 \phi}{\partial x_3 \partial x_2} = 0 \tag{2.7}$$

hold simultaneously. The general solution of equation (2.7) is

$$\rho\phi = -\varepsilon x_3 + \psi(x_1, x_2), \tag{2.8}$$

ε being a constant and ψ an arbitrary function of the arguments shown. From equations (2.2) and (2.3) one then finds

$$\frac{\partial}{\partial x_3}(T_{33} - \rho v) = \varepsilon \tag{2.9}$$

where T_{33} is the stress normal to R. The constant ε is thereby identified as the specific driving force in the flow.

The determinate stress $\mathbf{S}$ is computed from equations (2.4)–(2.6) whence there follows from the equations of motion (2.1) together with equation (2.8)

$$\frac{\partial}{\partial x_1}\left\{(4\alpha_1 + \alpha_2)\left(\frac{\partial v}{\partial x_1}\right)^2\right\} + \frac{\partial}{\partial x_2} \times \left\{(4\alpha_1 + \alpha_2)\frac{\partial v}{\partial x_1}\frac{\partial v}{\partial x_2}\right\} = 4\frac{\partial \psi}{\partial x_1}, \tag{2.10}$$

$$\frac{\partial}{\partial x_1}\left\{(4\alpha_1 + \alpha_2)\frac{\partial v}{\partial x_1}\frac{\partial v}{\partial x_2}\right\} + \frac{\partial}{\partial x_2} \times \left\{(4\alpha_1 + \alpha_2)\left(\frac{\partial v}{\partial x_2}\right)^2\right\} = 4\frac{\partial \psi}{\partial x_2}, \tag{2.11}$$

$$\frac{\partial}{\partial x_1}\left[\mu\frac{\partial v}{\partial x_1} + (\beta_2 + \beta_3)\frac{\partial v}{\partial x_2}\left\{\left(\frac{\partial v}{\partial x_1}\right)^2 + \left(\frac{\partial v}{\partial x_2}\right)^2\right\}\right] + \frac{\partial}{\partial x_2}\left[\mu\frac{\partial v}{\partial x_2} + (\beta_2 + \beta_3)\frac{\partial v}{\partial x_2} \times \left\{\left(\frac{\partial v}{\partial x_1}\right)^2 + \left(\frac{\partial v}{\partial x_2}\right)^2\right\}\right] = -2\varepsilon. \tag{2.12}$$

In conjunction with equations (2.10)–(2.12) one assumes the following expansions appropriate to a third-order fluid

$$v(x_1, x_2) = \sum_{k=1}^{3} \varepsilon^k v_k(x_1, x_2);$$

$$\psi(x_1, x_2) = \sum_{k=1}^{3} \varepsilon^k \psi_k(x_1, x_2) \quad (2.13)$$

in which terms of order four and higher in ε are neglected. The equations (2.10) and (2.11) can be regarded as determining the potential $\psi(x_1, x_2)$ once the velocity field has been obtained from (2.12). The substitution of the first of equations (2.13) into (2.12) leads to

$$\nabla^2 v_1 = -\frac{2}{\mu}; \qquad \nabla^2 v_2 = 0;$$

$$\nabla^2 v_3 + \left(\frac{\beta_2+\beta_3}{\mu}\right)\left[\nabla^2 v_1 \left\{\left(\frac{\partial v_1}{\partial x_1}\right)^2 + \left(\frac{\partial v_1}{\partial x_2}\right)^2\right\} + \quad (2.14)\right.$$

$$+ 4\frac{\partial v_1}{\partial x_1}\frac{\partial v_1}{\partial x_2}\frac{\partial^2 v_1}{\partial x_1 \partial x_2}$$

$$\left. + 2\left\{\left(\frac{\partial v_1}{\partial x_1}\right)^2 \frac{\partial^2 v_1}{\partial x_1^2} + \left(\frac{\partial v_1}{\partial x_2}\right)^2 \frac{\partial^2 v_1}{\partial x_2^2}\right\}\right] = 0$$

where ∇^2 is the Laplacian with respect to x_1 and x_2. In view of the boundary condition in equation (2.6) it is clear from the above that $v_2 \equiv 0$. Following a method used in [9] the solutions for v_1 and v_3 in (2.14) have been achieved as expansions of orthonormal functions which are complete on a rectangle and satisfy (2.6). The analysis is protracted but nonetheless straightforward. For a channel of aspect ratio $\gamma = a/b$ what emerges is

$$v_1(x_1, x_2) = \sum_{m,n} Ea_{mn} \times \sin\frac{m\pi(x_1 + a)}{2a}\sin\frac{n\pi(x_2 + b)}{2b} \quad (2.15)$$

in which

$$E = \frac{128a^2}{\mu\pi^4}; \quad a_{mn} = \frac{1}{mn(m^2 + \gamma^2 n^2)}, \quad (m, n = 1, 3, 5, \ldots) \quad (2.16)$$

and

$$v_3(x_1, x_2) = \sum_{m,n} Fb_{mn}\sin\frac{m\pi(x_1 + a)}{2a}\sin\frac{n\pi(x_2 + b)}{2b} \quad (2.17)$$

in which

$$F = \frac{2(128)^2 a^4\gamma^2(\beta_2 + \beta_3)}{\mu^4\pi^{10}}; \quad b_{mn} = \frac{\bar{b}_{mn}}{m^2 + \gamma^2 n^2}, \quad (m, n = 1, 3, 5, \ldots) \quad (2.18)$$

where

$$\bar{b}_{mn} = \sum_{k,p,q,r,s,t}\left[\frac{4}{pq}\delta^s_{kqm}\delta^t_{rpn}D_{kpqrst} - \frac{1}{pr}\left(\frac{3q}{\gamma^2 t} + \frac{t}{q}\right)\delta^s_{kqm}\hat{\delta}^t_{rpn}D_{kpsrqt} - \frac{1}{kq}\left(\frac{s}{p} + \frac{3\gamma^2 p}{s}\right)\delta^t_{rpn}\hat{\delta}^s_{kqm}D_{ktqrsp}\right]. \quad (2.19)$$

In equation (2.19) the symbols have the following meanings

$$\delta^s_{kqm} = \begin{cases} +1 & \text{if } s = |k+q-m| \text{ or } s = |k+m-q| \\ -1 & \text{if } s = |q+m-k| \text{ or } s = k+q+m \\ 0 & \text{otherwise} \end{cases}$$

$$\hat{\delta}^s_{kqm} = \begin{cases} \pm 1 & \text{if } s = |k+q-m| \text{ or } s = |k+m-q| \text{ or } s = |q+m-k|\text{, the } +(-) \text{ sign being chosen if a quantity inside } |\cdot| \text{ is } +(-) \\ -1 & \text{if } s = k+q+m \\ 0 & \text{otherwise} \end{cases}$$

$$D_{kpqrst} = [(k^2+\gamma^2p^2)(q^2+\gamma^2r^2)(s^2+\gamma^2t^2)]^{-1}.$$

Therefore the velocity field in a rectangular channel of aspect ratio γ is

$$\mathbf{v}(x_1, x_2) = \{\varepsilon v_1(x_1, x_2) + \varepsilon^3 v_3(x_1, x_2)\}\mathbf{k} \quad (2.20)$$

with v_1 and v_3 given by equations (2.15) and (2.17). The average velocity v_0 over any cross section of the pipe can be evaluated from equation (2.20). We find

$$v_0 = \frac{4\varepsilon}{\pi^2}\sum_{m,n}\frac{1}{mn}(Ea_{mn} + \varepsilon^2 Fb_{mn}). \quad (2.21)$$

3. THE TEMPERATURE FIELD

At $x_3 = 0$, the uniform temperature of the heated fluid is T_0, and T_1 ($< T_0$) is the constant wall temperature of the channel for $x_3 > 0$. If mechanical heating in the fluid is ignored the temperature field $T(x_1, x_2, x_3)$ satisfies the energy equation

$$\rho c_p v\frac{\partial T}{\partial x_3} = k\left(\frac{\partial^2 T}{\partial x_1^2} + \frac{\partial^2 T}{\partial x_2^2} + \frac{\partial^2 T}{\partial x_3^2}\right) \quad (3.1)$$

where v is the lineal flow (2.20). The neglect of mechanical heating is a valid approximation for sufficiently small Brinkman numbers. It is known, for example from Pearson [10], that this number can vary from 0 to ∞ for polymer melts, and for slow flows of melts in small bore channels the Brinkman number is small. Suitable dimensionless quantities are

$$\theta = \frac{T - T_1}{T_0 - T_1}, \quad x = \frac{(1+\gamma)(x_1 + a)}{4a}, \quad y = \frac{(1+\gamma)(x_2 + b)}{4a}, \quad z = \frac{x_3}{dPe} \quad (3.2)$$

$Pe = \rho d c_p v_0 k^{-1}$ being the Péclet number of the fluid, and $d = 4a(1+\gamma)^{-1}$ is a representative length—it is in fact equal to four times the area divided by the perimeter of the cross section. When the Péclet number is large, as is usually the case, e.g. in polymer processing situations (Pearson [10]), temperature gradients are small along streamlines and large across them. A standard approximation in this case is to retain only the transverse heat conduction contributions. The dimensionless temperature field is then found to satisfy

$$\nabla^2\theta = \frac{v}{v_0}\frac{\partial\theta}{\partial z}; \quad \theta(x,y,0) = 1;$$

$$\theta(x,y,z) = 0 \quad \text{on } x = 0, \tag{3.3}$$

$$x = \frac{1+\gamma}{2}, \quad y = 0, \quad y = \frac{1+\gamma}{2\gamma}.$$

From equations (2.20), (2.21) and (3.2) there follows

$$\frac{v}{v_0} = \frac{1}{K(\bar{\varepsilon})}\sum_{m,n}(a_{mn} + J\bar{\varepsilon}b_{mn})\sin\frac{2m\pi x}{1+\gamma}\sin\frac{2n\pi\gamma y}{1+\gamma} \tag{3.4}$$

where

$$\bar{\varepsilon} = \frac{16a^2(\beta_2+\beta_3)}{\mu^3(1+\gamma)^2}\varepsilon^2; \quad J = \frac{16\gamma^2(\gamma+1)^2}{\pi^6};$$

$$K(\bar{\varepsilon}) = \frac{4}{\pi^2}\sum_{m,n}\frac{1}{mn}(a_{mn} + \bar{\varepsilon}Jb_{mn}). \tag{3.5}$$

The nature of the dependance of v/v_0 on $\bar{\varepsilon}$ is clearly of paramount interest. Typically it has been demonstrated in [11] where profiles of v/v_0 for plane Poiseuille flow between parallel planes have been presented. As was pointed out there, negative $\bar{\varepsilon}$ values, which arise when the material combination $(\beta_2 + \beta_3)$ is negative, lead to diminished flow in the centre of the channel accompanied by a thinning of the region of retarded flow at the planes whereas for $\bar{\varepsilon}$ positive a core of increased flow is accompanied by a thickening of the retarded-flow region. We have mentioned earlier that the flow behaviour corresponding to the former case is representative of the polyacrylamide solutions used in [4] and [5]. Of course equation (3.4) reverts to the Newtonian profile if the dimensionless parameter $\bar{\varepsilon} = 0$, $\varepsilon > 0$.

The theory of the differential equation (3.3) is comprehensively treated by Courant and Hilbert [12]. In the present instance its solution is

$$\theta(x,y,z) = \sum_{m,n} A_{mn}\phi_{mn}(x,y)e^{-\lambda_{mn}z};$$

$$1 = \sum_{m,n} A_{mn}\phi_{mn}(x,y) \tag{3.6}$$

where

$$\phi_{mn}(x,y) = \sum_{p,q} C_{pq}^{(mn)}\sin\frac{2p\pi x}{1+\gamma}\sin\frac{2q\pi\gamma y}{1+\gamma} \tag{3.7}$$

and $p, q, m, n = 1, 3, 5, \ldots$. The coefficients $C_{pq}^{(mn)}$ appropriate to the eigensolution ϕ_{mn} satisfy the following set of homogeneous, algebraic equations,

$$M_{pq}C_{pq}^{(mn)} = \Lambda_{mn}\sum_{p,q,r,s,t,u} \times N_{pqrstu}C_{rs}^{(mn)}(a_{tu} + J\bar{\varepsilon}b_{tu}) \tag{3.8}$$

where

$$M_{pq} = p^2 + \gamma^2 q^2,$$

$$\Lambda_{mn} = \frac{(1+\gamma)^2}{\pi^4 K(\bar{\varepsilon})}\lambda_{mn},$$

$$N_{pqrstu} = rs\left\{\frac{1}{r^2-(t-p)^2} - \frac{1}{r^2-(t+p)^2}\right\} \times\left\{\frac{1}{s^2-(u-q)^2} - \frac{1}{s^2-(u+q)^2}\right\}$$

for $p, q, r, s, t, u = 1, 3, 5, \ldots$. We have used an ICL1906S computer to solve the equations (3.8). In the case of a square channel we have evaluated $\{\lambda_{mn}, C_{pq}^{(mn)}\}$ for $(mm) = (1,1), (1,3), (3,3)$ corresponding to $\bar{\varepsilon} = -1.0$, 0, 1.0 and the numerical values of the eigenvalues are displayed in Table 1. For a variety of rectangular channels and various values of $\bar{\varepsilon}$ we have computed the dominant eigenvalues λ_{11} which have been used to calculate the asymptotic heat-transfer coefficient outside the thermal entrance region. We should mention that the results we have obtained for $\bar{\varepsilon} = 0$ corroborate those of [9] for a Newtonian fluid.

4. THERMAL RESULTS

Each ϕ_{mn} of equations (3.6) and (3.7) has an arbitrary amplitude which we choose, for convenience, so that

$$\int_R v\phi_{mn}^2\,dx\,dy = \int_R v\,dx\,dy. \tag{4.1}$$

Experimental measurements may be made on the basis of a mean mixed temperature of the fluid, that is, $\theta(x, y, z)$ averaged with respect to the local fluid velocity over any cross section R of the channel. We denote this quantity by $\theta_M(z)$ and it is given by

$$\theta_M(z) = \int_R v\theta\,dx\,dy \Big/ \int_R v\,dx\,dy = \sum_{mn} A_{mn}^2 e^{-\lambda_{mn}z}. \tag{4.2}$$

This last equation (4.2) is arrived at through use of equations (3.6), (3.7) and (3.4), and it is found that

$$A_{mn}^2 = \frac{\pi^2}{16}\frac{1}{K(\bar{\varepsilon})}\left[\sum_{p,q} C_{pq}^{(mn)}V_{pq}\right]^2 \Big/ \sum_{p,q,r,s,t,u} N_{pqrstu}C_{rs}^{(mn)}C_{pq}^{(mn)}V_{tu} \tag{4.3}$$

where the symbols on the RHS have been previously defined except

$$V_{pq} = a_{pq} + J\bar{\varepsilon}b_{pq}.$$

The local heat-transfer coefficient is the Nusselt number given by

$$Nu(z) = -\frac{d}{P\theta_M}\oint_{\partial R}\frac{\partial\theta}{\partial v}\,ds \tag{4.4}$$

in which d is the representative length in equation (3.2),

Table 1.

	$\bar{\varepsilon} = -1.0$		$\bar{\varepsilon} = 0$		$\bar{\varepsilon} = 1.0$	
	$\lambda_{1,1} = 12.169$ $\lambda_{1,3} = \lambda_{3,1} = 72.025$ $\lambda_{3,3} = 158.55$	$A_{1,1}^2 = 0.8019$ $A_{1,3}^2 = A_{3,1}^2 = 0.1054$ $A_{3,3}^2 = 0.0125$	$\lambda_{1,1} = 11.910$ $\lambda_{1,3} = \lambda_{3,1} = 71.083$ $\lambda_{3,3} = 157.89$	$A_{1,1}^2 = 0.8044$ $A_{1,3}^2 = A_{3,1}^2 = 0.1041$ $A_{3,3}^2 = 0.0143$	$\lambda_{1,1} = 11.531$ $\lambda_{1,3} = \lambda_{3,1} = 69.716$ $\lambda_{3,3} = 156.55$	$A_{1,1}^2 = 0.8087$ $A_{1,3}^2 = A_{3,1}^2 = 0.1020$ $A_{3,3}^2 = 0.0134$
z	$\theta_M(z)$	$Nu(z)$	$\theta_M(z)$	$Nu(z)$	$\theta_M(z)$	$Nu(z)$
0.05	0.442	3.237	0.449	3.174	0.461	3.080
0.10	0.238	3.052	0.245	2.988	0.256	2.894
0.15	0.129	3.043	0.135	2.978	0.143	2.883
0.20	0.070	3.042	0.074	2.978	0.081	2.883
0.25	0.038		0.041		0.045	
0.30	0.021		0.023		0.025	
∞	0	3.042	0	2.978	0	2.883

Table 2.

ε \ γ	1.0	2.0	4.0	8.0	16.0	∞	Circle
				$Nu(\infty)$ values			
1.0	2.883	3.343	4.442	5.567	6.380	7.430	3.566
0.6	2.925	3.366	4.442	5.579	6.408	7.478	3.606
0.3	2.953	3.380	4.441	5.587	6.427	7.511	3.633
0	2.978	3.392	4.440	5.594	6.444	7.541	3.657
−0.3	2.999	3.403	4.439	5.600	6.460	7.568	3.679
−0.6	3.019	3.413	4.438	5.605	6.474	7.594	3.699
−1.0	3.042	3.424	4.436	5.611	6.491	7.625	3.724
−25.0	3.315	3.543	4.400	5.679	6.714	8.074	4.066
−50.0	3.347	3.555	4.394	5.686	6.744	8.141	4.144
$-\infty$	3.387	3.570	4.387	5.695	6.781	8.227	4.175

P is the perimeter of the cross section R with boundary ∂R, and ν is normal to ∂R. If we recall equations (3.3), (3.6) and (4.1) and apply the two dimensional Gauss theorem, the definition (4.4) reduces to

$$Nu(z) = \frac{1}{4\theta_M} \sum_{m,n} A_{mn}^2 \lambda_{mn} e^{-\lambda_{mn} z} \tag{4.5}$$

which, in view of (4.2), has the asymptotic value

$$Nu(\infty) = \lambda_{11}/4. \tag{4.6}$$

We have computed several values of A_{mn}^2 for a square channel taking $\varepsilon = -1.0, 0$ and 1.0. These are shown in Table 1 and have been used to calculate the functions $\theta_M(z)$ and $Nu(z)$ which are also displayed there. The trend of these functions is rather typical of rectangular channels in general and indicates, as was pointed out in [11], that those fluids which exhibit shear thinning velocity profiles afford enhanced Nusselt number values.

Values of $Nu(\infty)$ are displayed in Table 2 for varying ε values and rectangular channels of various aspect ratios γ. The case $\varepsilon = -\infty$ is hardly realistic physically but has been included to provide an upper bound on the heat-transfer coefficient for negative ε values. Also the results for the aspect ratio $\gamma = \infty$ have been taken from [11] for plane Poiseuille flow between two parallel plates, which is easily shown to be a limiting case of the current problem. The results for the circular tube which are included are taken from the same source. The results in Table 2 show that the asymptotic Nusselt number for any ε values is increased by some 150% for large γ values as compared with $\gamma = 1$ for the square. This corresponds to the figure of 90% obtained by Oliver and Karim [5] for increasingly flattened circular tubes. Where the channel is square and ε is negative, that is the fluid is shear thinning as described earlier, there is an enhancement in $Nu(\infty)$ of up to 14% over the Newtonian case $\varepsilon = 0$. Comparing the circle and the square it is clear that the former yields a greater heat-transfer coefficient than the latter for lineal flows whether the fluids exhibit normal stress effects or not. However, if we define the percentage increase in the asymptotic Nusselt number of rectangular channels compared to a circular tube as

$$I\% = \frac{Nu(\infty)_R - Nu(\infty)_C}{Nu(\infty)_C} \times 100$$

where subscripts R and C denote rectangle and circle, the values in Table 2 indicate general increases of some 100% for rectangles of large aspect ratio.

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CONVECTION THERMIQUE FORCEE POUR UN ECOULEMENT UNIDIRECTIONNEL DE FLUIDE NON NEWTONIEN DANS UN CANAL RECTANGULAIRE

Résumé—On étudie le flux thermique moyen transféré à la paroi d'un canal rectangulaire traversé par un fluide non Newtonien, chaud et incompressible. Le fluide, dit communément du troisième degré, est celui de Rivlin–Ericksen d'ordre le plus élevé, lequel s'écoule dans un écoulement secondaire transversal à l'intérieur d'un canal rectangulaire. Un coefficient de transfert thermique est évalué pour plusieurs valeurs du paramètre non Newtonien $\bar{\varepsilon}$ et pour différentes géométries depuis le carré jusqu'aux plans parallèles. Les résultats montrent un accroissement régulier du coefficient de transfert pour une valeur quelconque de $\bar{\varepsilon}$, depuis le carré jusqu'au rectangle allongé. On trouve aussi un accroissement du coefficient de transfert pour les fluides qui présentent de forts gradients de vitesse de déformation dans la région de paroi d'un canal rectangulaire quelconque.

Zusammenfassung—Untersucht wird die durchschnittliche Wärmeübertragung an die Wand eines rechtwinkligen Kanals, während eine erhitzte inkompressible, nicht-Newtonische Flussigkeit dadurch bewegt. Die Flüssigkeit ist die, die durchschnittlich 'dritten Grad' heisst, und ist die Rivlin–Ericksen Flüssigkeit höchster Ordnung, die geraldlinige Bewegung ohne zweitrangige Kreuzbewegung in einem rechtwinkligen Kanal hergeben kann. Ein Wärmeübertragungs-koeffizient ist für mehrere Werte eines nicht-Newtonischen Parameters, $\bar{\varepsilon}$, und für eine Klasse rechtwinkliger Geometrien vom Quadrat an einem Ende bis zu unbegrentzen, parallelen Flächen am anderen Ende berechnet worden. Die Ergebnisse zeigten wachsende Vergrösserung des Wärmeübertragungskoeffizienten für irgendeinen Wert von $\bar{\varepsilon}$, an Fortbewegung vom Quadrat nach engeren Rechtecken. Es wird auch eine Vergrösserung das Wärmübertragungskoeffizienten für jene Flüssigkeiten gefunden, die in der Nähe der Wand eines willkürlichen rechtwinkligen Kanals grosse Gradienten der Verzerrungsgeschwindigkeit aufweisen.

ВЫНУЖДЕННАЯ ТЕПЛОВАЯ КОНВЕКЦИЯ ПРИ ПРЯМОЛИНЕЙНОМ ТЕЧЕНИИ НЕНЬЮТОНОВСКОЙ ЖИДКОСТИ В ПРЯМОУГОЛЬНОМ КАНАЛЕ

Аннотация — Получены данные об интегральном теплообмене в прямоугольном канале при течении в нем нагретой несжимаемой жидкости. Использовалась модель среды «третьего порядка» типа Ривлина–Эриксена, когда при ламинарном течении отсутствуют вторичные поперечные токи. Рассчитан коэффициент теплообмена для нескольких значений параметра неньютоновости $\bar{\varepsilon}$ и ряда геометрий поперечника канала — от квадратного до плоскопараллельного. При любых значениях $\bar{\varepsilon}$ коэффициент теплообмена возрастает с изменением поперечника от квадрата до узкого прямоугольника и оказывается выше у жидкостей с более высокими скоростями сдвига в пристенной области.